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Generalization of matching extensions in graphs

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Abstract

Let G be a graph with vertex set $V(G)$. Let n, k and d be non-negative integers such that $n+2k+d \leqslant |V(G)|-2$ and $|V(G)|-n-d$ is even. A matching which covers exactly $|V(G)|-d$ vertices of G is called a *defect-d matching* of G. If when deleting any n vertices of G the remaining subgraph contains a matching of k edges and every k-matching can be extended to a defect-d matching, then G is called a (n,k,d)-*graph*. In this paper a characterization of (n,k,d)-graphs is given and several properties (such as connectivity, minimum degree, hierarchy, etc.) of (n,k,d)-graphs are investigated. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The graphs considered in this paper will be finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *matching* M of G is a subset of $E(G)$ such that any two edges of M have no vertices in common. A matching of k edges is called a k-*matching*. Let d be a non-negative integer. A matching is called a *defect-d matching* if it covers exactly $|V(G)|-d$ vertices of G. Clearly, a defect-0 matching is a perfect matching. A necessary and sufficient condition for a graph to have a defect-d matching was given by Berge [2]. Let S be a subset of $V(G)$. Denote by $G[S]$ the induced subgraph of G by S and we write $G-S$ for $G[V(G)\backslash S]$. The number of odd components of G is denoted by $o(G)$. Let M be a matching of G. If there is a

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matching M' of G such that $M \subseteq M'$, then we say that M can be extended to M' or M' is an extension of M. If each k-matching can be extended to a perfect matching, then G is called *k-extendable*. For convenience, a 0-extendable graph means a graph which has a perfect matching. A graph G is called *n-critical* if after deleting any n vertices the remaining subgraph of G has a perfect matching. In particular, a 2-critical graph is also called *bicritical*. The properties of bicriticability and k-extendability were studied extensively by Lovász and Plummer [6] and many results about these topics can be found in [6,8]. A characterization of n-critical graphs and their properties appears in [10]. In the present paper we introduce a new concept that is a combination and generalization of the above concepts.

Let G be a graph and let n, k and d be non-negative integers such that $n + 2k + d \leqslant |V(G)| - 2$ and $|V(G)| - n - d$ is even. If when deleting any n vertices from G the remaining subgraph of G contains a k-matching and each k-matching in the subgraph can be extended to a defect-d matching, then G is called a *(n, k, d)-graph*.

Clearly, a graph is a $(0, 0, d)$-graph if and only if it has a defect-d matching and a graph is a $(0, 0, 0)$-graph if and only if it has a perfect matching. Furthermore, $(0, k, 0)$-graphs are the same as k-extendable graphs [7] and $(n, 0, 0,)$-graphs are exactly n-critical graphs [10].

For example, Petersen Graph is a $(0, 1, 0)$-graph but not a $(0, 2, 0)$-graph, a $(1, 1, 1)$-graph but not a $(1, 2, 1)$-graph and a $(2, 1, 2)$-graph but not a $(2, 1, 0)$-graph. A complete graph K_{2m} is a $(2r, m - r - 1, 0)$-graph $(0 \leqslant r < m)$ and a $(2r - 1, m - r - 1, 1)$-graph $(1 \leqslant r < m)$.

In this paper a characterization of (n, k, d)-graphs is given and certain properties of (n, k, d)-graphs are discussed. In particular, the connectivity, minimum degree and hierarchy of these graphs are studied. Notation and definitions not given in this paper can be found in [3].

To conclude this section we present a few known results as lemmas.

Lemma 1.1 (Berge [2]). *Let G be a graph and let d be an integer such that $0 \leqslant d \leqslant |V(G)|$ and $|V(G)| \equiv d \pmod 2$. Then G has a defect-d matching if and only if for any $S \subseteq V(G)$*

$$o(G - S) \leqslant |S| + d.$$

Lemma 1.2 (Plummer [7]). *Let G be a connected graph and k be an integer such that $1 \leqslant k \leqslant |V(G)|/2 - 1$. If G is k-extendable, then G is $(k + 1)$-connected.*

Lemma 1.3 (Liu and Yu [4]). *Let G be a graph and n be an integer such that $1 \leqslant n \leqslant |V(G)| - 2$ and $n \equiv |V(G)| \pmod 2$. If G is n-critical, then G is n-connected.*

Note that an n-critical graph is always connected where $n \geqslant 1$.

Lemma 1.4 (Plummer [8]). *If graph G is connected and k-extendable where $k \geqslant 1$, then for any edge e of $G, G - e$ is $(k - 1)$-extendable.*

Lemma 1.5 (Yu [9]). *If G is a connected nonbipartite k-extendable graph where $k \geqslant 1$, then $G \cup e$ is $(k-1)$-extendable for any edge $e \notin E(G)$.*

2. Characterization of (*n, k, d*)-graphs

First we give a characterization of (n,k,d)-graphs. Then we show that some known results can be deduced from this characterization.

From now on we assume that n, k, and d are non-negative integers such that $n + 2k + d \leqslant |V(G)| - 2$ and $|V(G)| - n - d \equiv 0 \pmod 2$ for a graph G.

Theorem 2.1. *A graph G is a (n,k,d)-graph if and only if the following conditions hold*:

(i) *For any $S \subseteq V(G)$ and $|S| \geqslant n$, then*

$$o(G-S) \leqslant |S| - n + d.$$

(ii) *For any $S \subseteq V(G)$ such that $|S| \geqslant n + 2k$ and $G[S]$ contains a k-matching*,

$$o(G-S) \leqslant |S| - n - 2k + d.$$

Proof. Suppose that G is a (n,k,d)-graph. Let S be a subset of $V(G)$ such that $|S| \geqslant n$. Let $U \subseteq S$ and $|U| = n$. By the definition of (n,k,d)-graph, $G' = G - U$ has a defect-d matching. By Lemma 1.1, for $S' = S \backslash U$ we have

$$o(G' - S') \leqslant |S'| + d.$$

Thus

$$o(G-S) = o(G'-S') \leqslant |S'| + d = |S| - n + d,$$

that is, condition (i) holds.

Let $S \subseteq V(G)$, $|S| \geqslant n + 2k$ such that $G[S]$ contains a k-matching M. Let $V(M) \subseteq S_1 \subseteq S$ and $|S_1| = n + 2k$. Set $S' = S \backslash S_1$. Since G is a (n,k,d)-graph, $G' = G - S_1$ has a defect-d matching. By Lemma 1.1, $o(G' - S') \leqslant |S'| + d$ or

$$o(G-S) = o(G'-S') \leqslant |S'| + d = |S| - n - 2k + d.$$

Thus condition (ii) holds.

Conversely, suppose that both conditions (i) and (ii) hold. For any $U \subseteq V(G)$ with $|U| = n$, we show that $G' = G - U$ has a k-matching. For any $S' \subseteq V(G')$, let $S = U \cup S'$. Then $|S| \geqslant n$. By condition (i),

$$o(G-S) \leqslant |S| - n + d.$$

Thus

$$o(G'-S') = o(G-S) \leqslant |S| - n + d = |S'| + d.$$

By Lemma 1.1, G' has a defect-d matching. Since $n + 2k + d \leqslant |V(G)| - 2, G'$ contains a k-matching. Next, we show that any k-matching M of G' can be extended to a defect-d

matching of G'. Set $S_1 = U \cup V(M)$ and $G'' = G - S_1$. Then M can be extended to a defect-d matching of G' if and only if G'' has a defect-d matching. For any $S'' \subseteq V(G'')$, let $S_2 = S_1 \cup S''$. Then $|S_2| \geqslant n + 2k$ and $G[S_2]$ contains a k-matching. By condition (ii),

$$o(G - S_2) \leqslant |S_2| - n - 2k + d$$

or

$$o(G'' - S'') = o(G - S_2) \leqslant |S_2| - n - 2k + d = |S''| + d.$$

By Lemma 1.1, G'' has a defect-d matching. Therefore G is a (n,k,d)-graph. □

Setting $k = d = 0$ in Theorem 2.1, we obtain the following result first obtained by Yu [10].

Corollary 2.2 (Yu [10]). *Let G be a graph and n a non-negative integer such that $n \leqslant |V(G)| - 2$ and $|V(G)| \equiv n \pmod 2$. Then G is n-critical if and only if for any $S \subseteq V(G)$ and $|S| \geqslant n$,*

$$o(G - S) \leqslant |S| - n.$$

Setting $n = d = 0$ in Theorem 2.1, we obtain a characterization of k-extendable graphs below which was proven in [5,10].

Corollary 2.3 (Lou [5] and Yu [10]). *Let G be a graph and k a non-negative integer such that $2k \leqslant |V(G)| - 2$. Then G is k-extendable if and only if for any $S \subseteq V(G)$,*

$$o(G - S) \leqslant |S|$$

and for any $S \subseteq V(G)$ such that $G[S]$ contains a k-matching,

$$o(G - S) \leqslant |S| - 2k.$$

3. Properties of (n, k, d)-graphs

From Theorem 2.1 it is easy to see that a (n,k,d)-graph is also a (n,k,d')-graph where $d < d' < |V(G)| - n - 2k$ and $d' \equiv d \pmod 2$. In this section we show that every (n,k,d)-graph is also a (n',k',d)-graph where $0 \leqslant n' < n$, $0 \leqslant k' < k$ and $n' \equiv n \pmod 2$. We also consider the connectivity and minimum degree of $(n,k,0)$-graphs. Furthermore, we investigate the effect on a (n,k,d)-graph by deleting or adding an edge.

Theorem 3.1. *Every (n,k,d)-graph G is also a (n',k',d)-graph where $0 \leqslant n' < n$, $0 \leqslant k' < k$ and $n' \equiv n \pmod 2$.*

Proof. To prove the theorem, we need only to prove that (a) and (b) hold:

(a) When $n \geqslant 2$, G is a $(n-2,k,d)$-graph.
(b) When $k \geqslant 1$, G is a $(n,k-1,d)$-graph.

First we prove (a). Let $S \subseteq V(G)$ and $|S| \geqslant n-2$. We show that conditions (i) and (ii) in Theorem 2.1 hold. If $|S| \geqslant n$, then by Theorem 2.1 we have

$$o(G-S) \leqslant |S|-n+d < |S|-(n-2)+d.$$

We consider $|S|=n-1$. Since G is a (n,k,d)-graph and $n+2k \leqslant |V(G)|-d-2$, $G-S$ has at least one edge. Thus there is a component C of $G-S$ with at least two vertices. It is easy to see that we can always choose a vertex x in C such that $C-\{x\}$ is connected. Set $S'=S\cup\{x\}$. Clearly, when C is even, $o(G-S) \leqslant o(G-S')$; when C is odd, $o(G-S)=o(G-S')+1$. Thus

$$\begin{aligned} o(G-S) &\leqslant o(G-S')+1 \leqslant |S'|-n+d+1 \\ &= |S|+1-n+d+1=|S|-(n-2)+d. \end{aligned}$$

Now assume $|S|=n-2$. Since G is a (n,k,d)-graph and $n+2k \leqslant |V(G)|-d-2$, $G-S$ has at least one edge $e=xy$. Set $S''=S\cup\{x,y\}$. We have

$$o(G-S) \leqslant o(G-S'') \leqslant |S''|-n+D=|S|-(n-2)+d.$$

Thus condition (i) of Theorem 2.1 holds. Similarly, we can prove that condition (ii) of Theorem 2.1 holds for $S\subseteq V(G)$, $|S| \geqslant n-2+2k$ and $G[S]$ contains a k-matching. Thus by Theorem 2.1, G is a $(n-2,k,d)$-graph.

Now we prove that (b) holds. When $k=1$, it is clear that (b) is true. We assume $k \geqslant 2$. Let U be any subset of $V(G)$ with $|U|=n$. Set $G'=G-U$. Since G is a (n,k,d)-graph, G' has a $(k-1)$-matching. Let M_1 be any $(k-1)$-matching of G'. Let M be a defect-d matching of G'. Since $n+2k \leqslant |V(G)|-d-2$, it follows $|M| \geqslant |M_1|+2$. Consider the symmetric difference $M \triangle M_1$ of M and M_1. Then there must be a component P of $M \triangle M_1$ which is an alternating path of odd length such that both of the two end edges of P are in M. Thus $M_2=M_1 \triangle P$ is a k-matching. By the assumption M_2 can be extended to a defect-d matching M' of G'. Since $|M'| \geqslant |M_2|+1$, there is an edge $e\in M'\backslash M_2$. Thus $M_1\cup\{e\}$ is a k-matching in G'. By the definition of (n,k,d)-graph, $M_1\cup\{e\}$ can be extended to a defect-d matching of G', i.e., M_1 can be extended to a defect-d matching of G'. Hence (b) holds and so G is a $(n,k-1,d)$-graph. □

Set $d=n=0$. The following result from Plummer [7] can be deduced from Theorem 3.1.

Corollary 3.2 (Plummer [7]). *For $k \geqslant 1$ if a graph G is k-extendable, then it is also $(k-1)$-extendable.*

Setting $d=k=0$ we immediately obtain the following result.

Corollary 3.3 (Yu [10]). *For $n \geqslant 2$ if a graph G is n-critical, then it is also $(n-2)$-critical.*

Next we will focus our attention on the properties of $(n,k,0)$-graphs. At first, we give the following observation which can be seen easily from the definition of $(n,k,0)$-graph.

Observation. A graph G is a $(n,k,0)$-graph if and only if for any $S\subset V(G)$ with $|S|=n$, $G-S$ is k-extendable.

Obviously, a bipartite graph with an unbalanced bipartition cannot have a perfect matching. Thus, a $(n,k,0)$-graph with $n\geqslant 1$ cannot be bipartite. Following this argument, we easily obtain the following.

Proposition 3.4. *If G is a $(n,k,0)$-graph, then G does not have an induced bipartite subgraph with more than $|V(G)|-n$ vertices.*

Since $2k$-critical graphs must be k-extendable, it is not hard to obtain the next result.

Proposition 3.5. *Let G be a $(n,k,0)$-graph. Then it is also a $(n-2,k+1,0)$-graph.*

Applying Proposition 3.5 repeatedly, we obtain the following relationship between $(n,k,0)$-graphs and m-extendable graphs.

Corollary 3.6. *A $(n,k,0)$-graph of even order is $(k+\lfloor n/2\rfloor)$-extendable.*

The converse of Proposition 3.5 is given next.

Theorem 3.7. *If G is a $(n,k,0)$-graph and $n\geqslant 1$, $k\geqslant 2$, then G is a $(n+2, k-2,0)$-graph.*

Proof. Since $n\geqslant 1$, G is a nonbipartite graph.

Suppose that G is not a $(n+2,k-2,0)$-graph. Then there exists a $(k-2)$-matching M' (denote $V(M')$ by S') and a vertex set S'' of order $n+2$ such that $S'\cap S''=\emptyset$ and $G-(S'\cup S'')$ has no perfect matching.

Claim. *S'' is an independent set.*

Otherwise, let $e=uv$ be an edge in $G[S'']$, then $M'\cup e$ and $S''-\{u,v\}$ cannot be extended to a perfect matching in G. That is, G is not a $(n,k-1,0)$-graph. But, since G is a $(n,k,0)$-graph, by Theorem 3.1, G is a $(n,k-1,0)$-graph, a contradiction.

Let x and y be two vertices of S'' and denote $S''-\{x,y\}$ by S_1. Then $G-S_1$ is a nonbipartite graph. From the observation above, $G-S_1$ is k-extendable. Thus $(G-S_1)\cup\{xy\}$ is still nonbipartite and $(k-1)$-extendable by Lemma 1.5. Therefore, there exists a perfect matching containing $M'\cup\{xy\}$ in $G-S_1$ and hence $G-(S'\cup S'')$ has a perfect matching, a contradiction. □

Theorem 3.8. *Every connected $(n,k,0)$-graph G is $(n+k+1)$-connected where $k \geqslant 1$.*

Proof. By contradiction. If the connectivity of G is less than $n+k+1$, let S be a cut set of G with minimum cardinality, then $|S| \leqslant n+k$. Choose $S' \subset S$ with $|S'| = n$. Then, by the Observation above, $G - S'$ is a k-extendable graph and thus its connectivity is at least $k+1$. But $S - S'$ is a cut set of $G - S'$ and its cardinality is less than $k+1$, a contradiction. □

Note that when $n \geqslant 1$, a $(n,0,0)$-graph is n-connected by Lemma 1.3. But when $d > 0$, the conclusion of Theorem 3.8 is not true since the connectivity of a connected (n,k,d)-graph G may be only 1. For example, let G be a connected graph with the set of vertices $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_d\}$ and the set of edges $\{v_i v_j \mid i,\ j = 1,2,\ldots,m$ and $i \neq j\} \cup \{v_1u_1, v_1u_2, \ldots, v_1u_d\}$, where $m \equiv n \pmod 2$ and $m \geqslant n + 2k + 2$. Then G is a graph consist of complete graph with d leaves. It is easy to see that G is a (n,k,d)-graph, but the connectivity of G is only 1.

Next, we investigate the effect on matching extension by deleting or adding an edge to a graph. At first, we study the effect of deleting an edge from $(n,k,0)$-graphs.

Let $S \subseteq V(G)$ and C be an even component of $G - S$. If e is a cut edge of C such that both components of $C - e$ are odd, then e is called a *key cut edge* of $G - S$. It is not difficult to see that when e is a key cut edge of $G - S$, then $o(G-e-S) = o(G-S)+2$; otherwise, $o(G-e-S) = o(G-S)$.

Theorem 3.9. *Let G be a $(n,k,0)$-graph where $n \geqslant 2$. Then for any edge e of $G, G-e$ is a $(n-2,k,0)$-graph.*

Proof. Since $n \geqslant 2$, G must be connected. For any edge e of G let $G' = G - e$. We prove that G' satisfies conditions (i) and (ii) of Theorem 2.1 for $n-2$.

(a) Let $S \subseteq V(G')$ and $|S| \geqslant n-2$. By Theorem 3.1, G is a $(n-2,k,0)$-graph. Theorem 2.1 then implies that $o(G-S) \leqslant |S| - (n-2)$. When e is not a key cut edge of $G - S$, we have

$$o(G' - S) = o(G - S) \leqslant |S| - (n-2).$$

Now we assume that e is a key cut edge of $G - S$. If $o(G-S) < |S| - (n-2)$, since $o(G-S) \equiv |S| - (n-2) \pmod 2$, we have $o(G-S) \leqslant |S| - (n-2) - 2$. Thus

$$o(G' - S) = o(G - S) + 2 \leqslant |S| - (n-2).$$

Now we consider

$$o(G - S) = |S| - (n-2). \tag{3.1}$$

We show that (3.1) is impossible. Since G is a $(n,k,0)$-graph and $G - S$ has an even component, from the proof of (a) in Theorem 3.1 we know that $|S| = n-2$ when (3.1) holds. Let $e = ux$ be a key cut edge of C, and let C_1 and C_2 be two components of $C - e$ where $u \in V(C_1)$ and $x \in V(C_2)$. By Theorem 3.1 and Lemma 1.3, G is n-connected.

Since $|S| = n - 2$, $G - S$ is connected. Thus $G - S$ has only one component C. Since $n + 2k \leqslant |V(G)| - 2$, we have $|V(C)| \geqslant 4$. We may assume that $|V(C_2)| \geqslant 2$. Choose a vertex $y \neq x$ from C_2. Let $S' = S \cup \{x, y\}$. Then C_1 is an odd component of $G - S'$. Thus

$$o(G - S') \geqslant o(G - S) + 1.$$

By (3.1)

$$o(G - S') \geqslant |S| - (n - 2) + 1 = |S'| - n + 1. \tag{3.2}$$

Since G is a $(n, k, 0)$-graph, by Theorem 2.1

$$o(G - S') \leqslant |S'| - n, \tag{3.3}$$

which contradicts (3.2) and thus (3.1) is impossible. Thus G' satisfies condition (i) of Theorem 2.1 for $n - 2$.

(b) Now we prove that G' satisfies conditions (ii) of Theorem 2.1 for $n - 2$. Let $S \subseteq V(G)$ such that $|S| \geqslant n - 2 + 2k$ and $G[S]$ contains a k-matching. By Theorem 3.1 G is a $(n - 2, k, 0)$-graph. By Theorem 2.1

$$o(G - S) \leqslant |S| - (n - 2) - 2k. \tag{3.4}$$

Repeating the argument in (a) we know that when e is not a key cut edge of $G - S$ or $|S| > n - 2 + 2k$,

$$o(G' - S) \leqslant |S| - (n - 2) - 2k.$$

Now we assume that $|S| = n - 2 + 2k$ and e is a cut edge of an even component of $G - S$ such that $C - e$ has two odd components C_1 and C_2. By (3.4)

$$o(G - S) = 0. \tag{3.5}$$

Thus every component of $G - S$ is even. Let $e = ux$ and $u \in V(C_1)$, $x \in V(C_2)$. If $|V(C)| \geqslant 4$, as in (a) we can get a contradiction. If $|V(C)| = 2$, since $n + 2k \leqslant |V(G)| - 2$, $|V(G - S)| \geqslant 4$. Thus $G - S$ has another even component C'. Take a vertex $y \in V(C')$. Set $S' = S \cup \{x, y\}$. Then

$$o(G - S') \geqslant 2. \tag{3.6}$$

But since G is a $(n, k, 0)$-graph,

$$o(G - S') \leqslant |S'| - n - 2k = 0$$

a contradiction. Hence condition (ii) of Theorem 2.1 holds.

By Theorem 2.1, G' is a $(n - 2, k, 0)$-graph. □

Note that Theorem 3.9 is not true for $d \neq 0$. Let G be a graph such that $G_1, G_2, \ldots, G_d$ and C are the $d + 1$ components of G, where G_i $(1 \leqslant i \leqslant d)$ is a complete graph with $2m + 1 \geqslant 3$ vertices and $E(C) = \{e\}$. Clearly G is a $(2, 1, d)$-graph. But $G - e$ is not a $(0, 1, d)$-graph.

Theorem 3.10. *Let G be a connected $(n,k,0)$-graph and $k \geqslant 1$. Then for any edge e of G, $G-e$ is a $(n,k-1,0)$-graph.*

Proof. When $n=0$, the theorem holds by Lemma 1.4. Now we consider $n \geqslant 1$. Let U be any set of n vertices of $G' = G-e$. Then $G-U$ is k-extendable. Let $e=uv$. If at least one of u and v are in U, clearly, $G'-U = G-U$ is k-extendable. By Corollary 3.2, $G'-U$ is also $(k-1)$-extendable. Now we assume that both of u and v are not in U. Thus e is an edge of $G-U$. Since G is $(n+k+1)$-connected, $G-U$ is connected. Lemma 1.4 yields that $G-U-e = G'-U$ is $(k-1)$-extendable. Therefore $G'=G-e$ is a $(n,k-1,0)$-graph. □.

It is easy to see that if G is not connected, the conclusion of Theorem 3.10 does not hold. An unanswered question remains: when do Theorems 3.9 and 3.10 hold for $d>0$?

Next, we investigate the effect of adding an edge to $(n,k,0)$-graphs.

Theorem 3.11. *Let G be a $(n,k,0)$-graph with $n,k \geqslant 1$. Then for any edge $e \notin E(G)$*

(a) *$G \cup e$ is a $(n,k-1,0)$-graph*;
(b) *$G \cup e$ is a $(n-2,k,0)$-graph.*

Proof. (a) Let S be any subset of $V(G)$ with $|S|=n$ and $M=\{e_1,e_2,\ldots,e_{k-1}\}$ a $(k-1)$-matching in $G-S$. Consider an edge $e=uv \notin E(G)$.

Case 1: $\{u,v\} \subseteq V(G-S)$. Since $G-S$ is a k-extendable graph and not a bipartite graph, $(G-S)\cup e$ is $(k-1)$-extendable graph by Lemma 1.5. Hence, M can be extended to a perfect matching of $G-S$.

Case 2: $\{u,v\} \nsubseteq V(G-S)$, say $u \in S$. Then $(G-S)\cup e = G-S$ is k-extendable and thus $(k-1)$-extendable. So S and M can be extended to a perfect matching of G.

(b) Let S' be any vertex subset of G with order $n-2$ and M a k-matching in $G-S'$. Let $S''=V(M)$ and $G'=G-(S'\cup S'')$. Consider $e=uv \notin E(G)$.

Case 1: $|\{u,v\}\cap(S'\cup S'')| \geqslant 1$. Since G is $(n-2,k,0)$-graph by Theorem 3.1, then $(G\cup e)-(S'\cup S'') = G-(S'\cup S'')$ has a perfect matching.

Case 2: $\{u,v\}\cap(S'\cup S'')=\emptyset$ or $\{u,v\}\subseteq V(G')$. Let $S_1 = S'\cup\{u,v\}$. Then $|S_1|=n$. Since G is a $(n,k,0)$-graph, $G-(S_1\cup S'') = (G\cup e)-(S_1\cup S'')$ has a perfect matching.

Therefore $G\cup e$ is a $(n-2,k,0)$-graph. □

Finally, we turn our attention to the minimum degree of $(n,k,0)$-graphs. In [1], Ananchuen and Caccetta discussed the minimum degree of k-extendable graphs and found a certain forbidden range for the minimum degrees. We present their result next.

Lemma 3.12 (Ananchuen and Caccetta [1]). *If G is a k-extendable graph on $2m$ vertices (where $1 \leqslant k \leqslant m-1$), then $k+1 \leqslant \delta(G) \leqslant m$ or $\delta(G) \geqslant 2k+1$.*

We will generalize the above results to $(n,k,0)$-graphs.

Theorem 3.13. *If G is a $(n,k,0)$-graph on v vertices, then $k+n+1 \leqslant \delta(G) \leqslant (v+n)/2$ or $\delta(G) \geqslant 2k+n+1$.*

Proof. From Theorem 3.8, G is $(k+n+1)$-connected and so its minimum degree is at least $k+n+1$.

Let u be a vertex of G having the minimum degree δ and $S \subseteq N_G(u)$ with $|S|=n$. Then $G-S$ is k-extendable and $\delta(G)=\delta_{G-S}(u)+n$. Let $\delta_{G-S}(u)=\delta'$. From Lemma 3.12, $\delta' \leqslant |V(G-S)|/2$ or $\delta' \geqslant 2k+1$. That is, $\delta=\delta'+n \leqslant (v+n)/2$ or $\delta=\delta'+n \geqslant 2k+n+1$. □

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